AM1.2, kolokwium 7.4.2022

4. (a) Wyznaczyć wielomian Taylora trzeciego stopnia w $x_0 = 0$ (wielomian Maclaurina) 5pkt dla funkcji

$$f(x) = e^{-\operatorname{arc\,tg} x}.$$

(b) Obliczyć granicę 5pkt

$$\lim_{x\to 0} \frac{\sin(2x) + \ln\left(\frac{1-x}{1+x}\right)}{\exp(-\operatorname{arc\,tg}\, x) - \frac{1}{1+x} + \frac{x^2}{2}}.$$

Odp:

(a) $e^{-\operatorname{arctg} x} = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$

(b) $\boxed{\sin(2x) + \ln\frac{1-x}{1+x} = -2x^3 + o(x^3)}$

bo

$$\begin{cases} \sin(2x) = 2x - \frac{4}{3}x^3 + o(x^3) \\ \ln\frac{1-x}{1+x} = \ln(1-x) - \ln(1+x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) = -2x - \frac{2}{3}x^3 + o(x^3) \end{cases}$$

$$M(x) = \left.\begin{array}{l} 1 - x + \frac{x^2}{2} + \frac{x^3}{6} \\ -\left(1 - x + x^2 - x^3\right) \\ + x^2/2 \end{array}\right\} = \frac{7}{6}x^3 + o(x^3)$$

Ans: $\frac{-2}{7/6} = -\frac{12}{7}$

II sposób (reg. de l'Hospitala)

$L(x) = \sin(2x) + \ln\frac{1-x}{1+x} \Rightarrow L'(x) = 2\left(\frac{1}{x^2-1} + \cos(2x)\right)$, $L'(0) = 0$

$L''(x) = 2\frac{d}{dx}\left(\cos(2x) + \frac{1}{x^2-1}\right) = -\frac{2\cdot 2x}{(x^2-1)^2} - 2\cdot 2\sin(2x)$, $L''(0) = 0$

$L'''(x) = 2\cdot\frac{d}{dx}\left(-2\sin(2x) - \frac{2x}{(x^2-1)^2}\right) = \frac{6x^2 - 4(x^2-1)^3\cos(2x) + 2}{(x^2-1)^3}\cdot 2$

$L'''(0) = (4\cdot 1 + 2)/(-1) = -12$ $\left(L_1'''(0) = -8,\ L_2'''(0) = -4 \ (\text{od } \frac{1}{x^2-1})\right)$

$$\frac{d}{dx}\left(-\frac{1}{1+x}+\frac{x^2}{2}\right)=x+\frac{1}{(1+x)^2}$$

$$\frac{d}{dx}\left(x+\frac{1}{(1+x)^2}\right)=1-\frac{2}{(1+x)^3}$$

$$\frac{d}{dx}\left(1-\frac{2}{(1+x)^3}\right)=\frac{6}{(1+x)^4}$$. i jeszcze pochodna (ostatni x)

$M_1'''(0) = 6$, $M_2(x) = \exp(-\operatorname{arctg} x)$

$$\frac{d}{dx}\left(e^{-\tan^{-1}(x)}\right)=-\frac{e^{-\tan^{-1}(x)}}{x^2+1}$$

$$\frac{d}{dx}\left(-\frac{e^{-\tan^{-1}(x)}}{1+x^2}\right)=\frac{(2x+1)\,e^{-\tan^{-1}(x)}}{(1+x^2)^2}$$

$$M_2'''(x)=\frac{d}{dx}\left(\frac{(2x+1)\,e^{-\tan^{-1}(x)}}{(1+x^2)^2}\right)=\frac{(-6x^2-6x+1)\,e^{-\tan^{-1}(x)}}{(1+x^2)^3} \Rightarrow M_2'''(0)=\frac{1\cdot e^0}{1}=1$$

$$\Rightarrow M'''(0) = 6+1 = 7 \quad a \quad \frac{L'''(0)}{M'''(0)} = \frac{-12}{7} \text{ (odpowiedź)}$$

Ocenianie

(a) 3 pkt za znalezienie rozwinięcia rzędu 2 $\left(1 - x + \frac{x^2}{2} + o(x^2)\right)$

(b) 1 pkt $\sin 2x$

1 pkt $\frac{1}{1+x}$

2 pkt : $\ln \frac{1-x}{1+x}$